Projection-induced channels and additivity violation of minimum output entropy

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Can entanglement make two channels quieter together?

$$S^{\min}(\Phi) := \min_{\rho} S(\Phi(\rho)), \qquad S^{\min}(\Phi_1 \otimes \Phi_2) \leq S^{\min}(\Phi_1) + S^{\min}(\Phi_2).$$

Classical channels: equality

Entropy is concave, so it is enough to test deterministic inputs. A deterministic input to two channels is automatically a product pair.

Quantum channels: strict inequality

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Why this matters

This is a universal additivity question governs whether the HSW classical capacity is single-letter and whether entanglement of formation is additive (Shor, 2004, Commun. Math. Phys.)
See also Prof. Winter's talk today.

Overview of the main results of this talk

Additivity question for minimum-output-entropy

For $p \geq 0$, $S_p(\rho) = \frac{1}{1-p} \log \mathrm{Tr}(\rho^p)$ is the p-Renyi entropy, do we have

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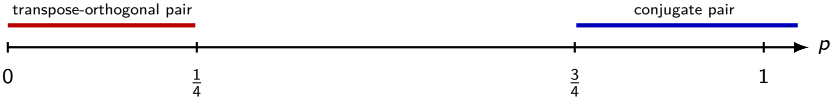
Theorem 1

For any $p > 3/4$ or $0 \leq p < 1/4$, there exists a pair of finite dimensional quantum channels (Φ, Ψ) such that

$$S_p^{\min}(\Phi \otimes \Psi) < S_p^{\min}(\Phi) + S_p^{\min}(\Psi).$$

- Previous unknown regime is $(0.12, 1)$, we reduce it to $[1/4, 3/4]$.
- Our approach indicates that $p = 1$ (von Neumann entropy case) is not a singular point.



Theorem 2

For any $p > 3/4$ or $0 \leq p < 1/4$, there exist explicit constructions of quantum channels (Φ, Ψ) from infinite dimensional system (with explicit Fock-type basis) to finite dimensional systems, such that

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- $(\mathcal{M}, \tau)$ is a tracial von Neumann algebra, $\Phi : S(\mathcal{M}) \to \mathcal{D}_k$ is a quantum channel, if it is CPTP.
- We use maximal tensor product of two von Neumann algebras. Minimum tensor product is equivalent to finite-dimensional approximable.
- The von Neumann algebra we construct is a corner of a noninjective type II_1 factor, which enters the zone where finite-dimensional approximation is tricky— Connes 1976, Ann. of Math.

Suppose P is a projection on a infinite dimensional system (with some technical assumptions), then the quantum channel maps an infinite dimensional state σ to a state on a k-level system:

$$\sum_{i,j=1}^{k} \big(\langle i|P\sigma P|j\rangle\big)|i\rangle\langle j|.$$

Explicit construction at a glance

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$$\sum_{i,j=1}^{k} \big(\langle i|P\sigma P|j\rangle\big)|i\rangle\langle j|.$$

- This simple construction is the free limit of the Haar random examples, studied in Auburn-Szarek-Werner 2011, Commun. Math. Phys; Belinschi-Collins-Nechita 2016, Commun. Math. Phys.
- This construction dates back to G. Lüders in 1951: Über die Zustandsänderung durch den Meßprozeß. Lüders introduced it as "a measurement followed by k-level selection" to explain state-change due to measurement of degenerate quantities, see Eq. (7) in a English translation of the paper https://arxiv.org/pdf/quant-ph/0403007.

Where the magic happens?

A long standing problem in random matrix theory

Pisier in the 90s asked: suppose $U_1^{n\times n}, \cdots U_k^{n\times n}$ are i.i.d. Haar random unitaries with size n, what is

$$\lim_{n\to\infty} \| \sum_{j=1}^{k} U_j^{n\times n} \|_{op} =?$$

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Resolved by Collins-Male 2014, this result motivated a lot of recent progress of free probability theory. In particular, we derive the following result using the technology developped in recent years:

Lemma

Suppose $P_n = \sum_{i,j=1}^{k} P_n^{ij} \otimes |i\rangle\langle j|$ is a Haar random projection on $\mathbb{C}^n \otimes \mathbb{C}^k$ with rank d_n and $d_n/kn \to t \in (0,1)$, what is

$$\lim_{n\to\infty} \lambda_{\max}(\sum_{i=1}^{k} a_i P_n^{ii}) = ?$$

for any real vector $a \in \mathbb{R}^k$.

Finite-dimensional random projection channels

Quantum channels induced by bipartite projections

Let $P = P_{AB}$ be a projection on $\mathcal{H}_A \otimes \mathcal{H}_B$, with $\dim \mathcal{H}_A = n$ and $\dim \mathcal{H}_B = k$. Put

$$P_A := \operatorname{Tr}_B P.$$

Whenever $P_A > 0$,

$$J_P = (P_A^{-1/2} \otimes I_k) P (P_A^{-1/2} \otimes I_k)$$

satisfies $\operatorname{Tr}_B J_P = I_A$ and is therefore the Choi matrix of a channel denoted as

$$\Phi_P(X) = \operatorname{Tr}_A \left[J_P (X^T \otimes I_k) \right].$$

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Haar-random projection-induced channel

Choose P_n Haar-random of rank d_n on $\mathbb{C}^n \otimes \mathbb{C}^k$ and assume

$$\frac{d_n}{kn} \longrightarrow t \in (\frac{1}{k^2}, 1).$$

Generically,

$$\mathrm{rank}(P_{A,n}) = \min\{n, k d_n\}.$$

Thus $t > 1/k^2$ is a clean asymptotic full-rank regime.

One channel works for both cases

Regime	Pair of channels	Correlated test input and output feature
High p	$\Phi_P \otimes \overline{\Phi}_P$: Hayden-Winter phenomenon	The maximally entangled input creates one exceptional Bell eigenvalue and a flat tail.
Low p	$\Phi_P \otimes \Phi_P^{\perp}$: Cubitt-Harrow-Leung-Montanaro-Winter phenomenon	A locally filtered Bell input kills one output direction, hence the joint output has rank at most $k^2 - 1$.

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Conventions

$$\overline{\Phi}_P(X) = \overline{\Phi_P(\overline{X})}, \qquad \Phi_P^{\perp} := \Phi_{I-P^T}.$$

Here $\Phi_P^{\perp}$ is the *transpose-orthogonal partner*.

Common proof strategy

Determine the one-channel output body exactly, then beat the entropy of a joint channel with one explicit joint input.

Digression: comparison to a well-known subspace construction

Suppose $P = P_{BE}$ is a projection on $\mathcal{H}_B \otimes \mathcal{H}_E \cong \mathbb{C}^k \otimes \mathbb{C}^d$. Denote

$$\mathcal{A} := P(\mathbb{M}_k \otimes \mathbb{M}_d)P, \quad \mathbf{1}_{\mathcal{A}} = P,$$

and a unital CP map

$$\Gamma : \mathbb{M}_k \to \mathcal{A}, \quad Y \mapsto P(Y \otimes I_d)P.$$

The adjoint channel is called the quantum channel induced by the subspace *Range*(P).

- The channel presented at the beginning is the limit by taking Haar random projection with rank $P = n$, $d_n/n \to t$ and send $n \to \infty$.
- This construction works perfectly for $p > 3/4$, it is not clear whether it works in the low p regime.

Haar randomness converges to a deterministic output body

We work with the projection-induced channel via Choi matrix. Recall that we choose P_n Haar-random of rank d_n on $\mathbb{C}^n \otimes \mathbb{C}^k$ and assume

$$\frac{d_n}{kn} \longrightarrow t \in (\frac{1}{k}, 1), \quad n = n_{in}, k = k_{out}.$$

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Theorem

For fixed $k \geq 2$ and $t \in (1/k^2, 1)$, the output state space of the channel $\Phi_n = \Phi_{P_n}$ converges almost surely to

$$\left\{ \frac{X}{\mathrm{Tr}(X)} : 0 \leq X \leq I_k, \mathrm{Tr}(f_t(X)) \leq \frac{1}{k} \right\}, \quad f_t(x) = \left(\sqrt{(1-t)x} - \sqrt{t(1-x)} \right)^2. \tag{1}$$

Consequently, for fixed $t \in (0, 1)$,

$$\lim_{n \to \infty} S_p^{\min}(\Phi_n) = \log k - \frac{2p(1-t)}{t\, k^2} + o(k^{-2}), \quad k \to \infty.$$

Conclude the proof

Let ψ_n^+ be the maximally entangled input and set

$$Z_n = (\Phi_{P_n} \otimes \overline{\Phi}_{P_n})(\psi_n^+).$$

Almost surely,

$$Z_n \longrightarrow r_{k,t}\psi_k^+ + (1 - r_{k,t})\frac{I_{k^2}}{k^2}, \quad r_{k,t} = \frac{k^2(1-t)}{(k^2-1)^2 t + 1 - t}.$$

$p > 3/4$ case

$$2S_p^{\min}(\Phi_n) \approx 2\log k - \frac{4p(1-t)}{t\,k^2} + o(k^{-2}),$$

$$S_p^{\min}(\Phi_n \otimes \overline{\Phi}_n) \leq 2\log k - \frac{A_p(t)}{k^2} + o(k^{-2}).$$

where

$$A_p(t) = \frac{t^{-p} - 1 - p(t^{-1} - 1)}{p - 1}.$$

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$p < 1/4$ case

$$2S_p^{\min}(\Phi_n) \approx 2\log k - \frac{4p}{k^2} + o(k^{-2}),$$

$$S_p^{\min}(\Phi_n \otimes \overline{\Phi}_n) \le \log(k^2 - 1) = 2\log k - \frac{1}{k^2} + o(k^{-2})$$

where we choose $t = 0.5$, and the last bound follows from Cubitt-Harrow-Leung-Montanaro-Winter 2008 Commun. Math. Phys.

Deterministic channel in infinite dimensional system

Haar projection gives two inequivalent channel recipes

Let P act on a bipartite space.

	Input system	Channel induced by P
Construction I	$\mathcal{H}_A$	Locally normalized Choi matrix $$J_P = (P_A^{-1/2} \otimes I)P(P_A^{-1/2} \otimes I).$$
Construction II	$\operatorname{Ran} P$	Random-subspace/Stinespring channel with Heisenberg adjoint $$\widetilde{\Gamma}_P(A) = P(I_A \otimes A)P\Big\vert_{\operatorname{Ran} P}.$$

We focus on the subspace channel for illustrational purpose. Recall P is a projection and

$$\sigma \mapsto \sum_{i,j=1}^{k} \big(\langle i|P\sigma P|j\rangle\big)|i\rangle\langle j|.$$

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Truly infinite dimensional phenomenon: no finite dimensional channel exists

Assume P has normalized trace $t \in (0,1)$ and some other technical properties, then the above channel from the infinite dimensional system to k-level system has an output state space

$$\{\rho \in \mathcal{D}_k : F(\rho, \frac{I_k}{k}) \geq 1 - t\}. \tag{2}$$

Explicit constructions in infinite dimensional systems

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Infinite dimensional Holevo-Werner-Hayden-Winter phenomenon

The above channel, tensoring with its complex conjugate, sends the "maximally entangle state" to an isotropic state, i.e., linear combination of maximally entangled state and maximally mixed state.

The maximal tensor product supplies an operator-algebraic Bell state

Infinite-dimensional Bell state

The infinite dimensional Bell state is defined by

$$\boxed{\omega_\Delta(a \otimes \overline{b}) := \tau(ab^*).}$$

This is the analogue of

$$\mathrm{Tr}\left[\psi_k^+(A \otimes \overline{B})\right] = \mathrm{tr}_k(AB^*).$$

The state has tracial marginals, but it is not itself a trace on the tensor product in infinite dimensions.

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Injectivity criterion

For a finite von Neumann algebra, continuity of the complete diagonal functional for the minimal norm is equivalent to injectivity/amenability (Connes 1976, Ann. of Math). The relevant free-product corner is noninjective, so ω_Δ is not a state on the fixed minimal tensor product.

Takeaways

1. We show Rényi-additivity violations for

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1. We show Rényi-additivity violations for
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using two explicit correlated input ansatz.

2. In infinite dimension case, the projection model has a fixed fidelity-type output body and an exact maximal-tensor isotropic witness. No finite dimensional channel has this property.

3. Removing randomness costs an infinite-dimensional commuting-operator input model, an explicit finite dimensional approximation is obstructed by non-existence of universal Connes embedding (MIP* = RE); thus an explicit finite-dimensional construction at $p = 1$ remains open. (This algebra is very special, so it is possible to get an approximation directly.)

Thank you for listening

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